

ECE 275A Homework # 1 Solutions

1. To solve this problem, we need to start with some known relationship between $\|x\|$, $\|y\|$, and $\langle x, y \rangle$, possibly with one or more free parameters to give us some flexibility in deriving the C-S inequality. We will discuss three proofs, the last one being the proof given in the textbook by Moon and Stirling.

First Proof. Perhaps the simplest parametric relationship we can envisage (having only one free parameter) is

$$0 \leq \|x + \beta y\|^2 = \langle x + \beta y, x + \beta y \rangle = \|x\|^2 + 2 \operatorname{Re} \beta \langle x, y \rangle + |\beta|^2 \|y\|^2. \quad (1)$$

Note that this relationship involves the squares of the norms of x and y , whereas the C-S inequality involves the norms themselves. Thus, in order to understand better where we are heading, let us rewrite the C-S inequality as,

$$\frac{|\langle x, y \rangle|^2}{\|y\|^2} \leq \|x\|^2. \quad (2)$$

We can now try to find a value for α which will place equation (1) into the form (2). Without too much work, one can show that the value,

$$\beta = -\frac{\overline{\langle x, y \rangle}}{\|y\|^2},$$

will do the job. Note the educated guessing involved in this proof, which is a standard proof given in many textbooks. However, in lieu of an educated guess, one can also ask what value of β makes the inequality shown above as tight as possible and see where that line of inquiry leads. Indeed, this latter tack is essentially the second proof described immediately below.

Second Proof. This is the optimization-based proof given in the Moon & Stirling text, which is based on solving,

$$\min_{\rho} \|x - \rho y\|^2.$$

Although the proof Moon & Stirling use is certainly valid *qua* proof, it is not one that is necessarily natural to use by a student at this stage of knowledge. Of course, one can ask the question as to which value of ρ gives the tightest bound, and the answer to this question, which is given by Moon & Stirling, happens to also show that the C-S inequality holds. Note that by asking for the specific value of ρ for which the always nonnegative quantity $\|x - \rho y\|^2$ is minimized, Moon & Stirling are effectively asking for the optimal projection of x onto the one-dimensional subspace spanned by y in the least-squares sense. This optimal choice of ρ happens to be given by $\rho = -\beta$, where β has the value we found in the first proof given above.

Third Proof. An alternative proof, one that involves no educated guessing or optimization, is to note that for all scalars α and β we have,

$$\begin{aligned} 0 \leq \|\alpha x + \beta y\|^2 &= \langle \alpha x + \beta y, \alpha x + \beta y \rangle \\ &= |\alpha|^2 \|x\|^2 + \bar{\alpha}\beta \langle x, y \rangle + \bar{\beta}\alpha \langle y, x \rangle + |\beta|^2 \|y\|^2 \\ &= \begin{pmatrix} \alpha \\ \beta \end{pmatrix}^H \begin{pmatrix} \|x\|^2 & \langle x, y \rangle \\ \langle y, x \rangle & \|y\|^2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \end{aligned}$$

which (because $\|x\|^2 \geq 0$) is true iff,¹

$$\det \begin{pmatrix} \|x\|^2 & \langle x, y \rangle \\ \langle y, x \rangle & \|y\|^2 \end{pmatrix} = \|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2 \geq 0,$$

which yields the C-S inequality. Note that if x and y are both nonzero and not collinear (i.e., are not proportional) we must have the strict inequality,

$$\|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2 > 0,$$

whereas if nonzero x and y are proportional, we must have strict equality,

$$\|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2 = 0.$$

2. Moon 2.1-4. The proof for both cases is essentially the same as \mathbf{R}^1 is just a specific instance of the general \mathbf{R}^n case. We have (take $\alpha = -1$ in equation (1) above),

$$\begin{aligned} \|x + y\|^2 &= \|x\|^2 + 2 \operatorname{Re} \langle x, y \rangle + \|y\|^2 \\ &\leq \|x\|^2 + 2 |\langle x, y \rangle| + \|y\|^2 \\ &\leq \|x\|^2 + 2 \|x\| \|y\| + \|y\|^2 \\ &\leq (\|x\| + \|y\|)^2. \end{aligned}$$

Taking the square root of both sides yields the triangle inequality.

3. Moon 2.12-57 and 2.12-58. All the desired properties of vector addition and scalar multiplication are inherited from the parent space.

Let v_1 and v_2 both belong to $\mathcal{V} \cap \mathcal{W}$. (Note that this means that v_1 and v_2 both belong to the subspace \mathcal{V} and both belong to the subspace \mathcal{W} .) Then for all scalars α_1 and α_2 , $\alpha_1 v_1 + \alpha_2 v_2 \in \mathcal{V}$, because \mathcal{V} is a subspace, and $\alpha_1 v_1 + \alpha_2 v_2 \in \mathcal{W}$, because \mathcal{W} is a subspace. Therefore $\alpha_1 v_1 + \alpha_2 v_2 \in \mathcal{V} \cap \mathcal{W}$, showing that we satisfy the required closure condition.

Let $v = v_1 + v_2$ and $w = w_1 + w_2$, where $v_1, w_1 \in \mathcal{V}$ and $v_2, w_2 \in \mathcal{W}$. The vectors v and w represent typical elements in the set $\mathcal{V} + \mathcal{W}$. For all scalars α and β , we have $\alpha v + \beta w = (\alpha v_1 + \beta w_1) + (\alpha v_2 + \beta w_2) \in \mathcal{V} + \mathcal{W}$, because $(\alpha v_1 + \beta w_1) \in \mathcal{V}$ and $(\alpha v_2 + \beta w_2) \in \mathcal{W}$ due to the fact that \mathcal{V} and \mathcal{W} are both subspaces.

¹That is, the matrix in the preceding quadratic form must be positive semidefinite, which is true iff its leading principal minors are nonnegative.

4. Moon 2.12-60. (a) Assume $\mathcal{V} \perp \mathcal{W}$ and $x \in \mathcal{V} \cap \mathcal{W}$. Then we must have $x \perp x$! This implies that $\langle x, x \rangle = 0$, which is true if and only if $x = 0$. Therefore \mathcal{V} and \mathcal{W} are disjoint. (Note that we have also shown that the only vector orthogonal to itself is the zero vector.) (b) Any simple counterexample will suffice.
5. Moon 2.12-73. The representation shown for the operator P is known as a *spectral representation*. (a) Note that by construction $P_i P_j = \delta_{i,j} P_j$ and therefore $PP_j = \lambda_j P_j$. Also note that $x \in \mathcal{V}_j$ iff $x = P_j x$. Therefore for $x \in \mathcal{V}_j$,

$$Px = PP_j x = \lambda_j P_j x = \lambda_j x.$$

(b) Let P be a projection operator onto a subspace \mathcal{V} along the subspace \mathcal{W} . (If P is an orthogonal projection operator, we take $\mathcal{W} = \mathcal{V}^\perp$.) then $Px = \lambda x$ for only two cases. Either $x \in \mathcal{V}$, in which case $Px = x$ and $\lambda = 1$, or $x \in \mathcal{W}$, in which case $Px = 0 \cdot x$ and $\lambda = 0$.

6. Let A be hermitian, $A^H = A$. (a) Note that $\overline{x^H Ax} = (x^H Ax)^H = x^H A^H x = x^H Ax$, and therefore $x^H Ax$ is real. Let x be any eigenvector of A with eigenvalue λ . Then from $Ax = \lambda x$ we obtain,

$$x^H Ax = \lambda x^H x = \lambda \|x\|^2. \quad (3)$$

Therefore λ must be real. (b) From equation (3). we see that if $x^H Ax$ is nonnegative, then the eigenvalue λ must be nonnegative. (c) If A has multiple eigenvectors for the same eigenvalue λ (in this case we say that these vectors span the *eigenspace* associated with λ), we can apply the Gram-Schmidt procedure to orthogonalize within this eigenspace. Now suppose that x_1 and x_2 are normalized eigenvectors associated with the two *distinct* eigenvalues λ_1 and λ_2 respectively. Then, using the fact that the eigenvalues are real,

$$\lambda_1 \langle x_1, x_2 \rangle = \lambda_1 x_1^H x_2 = (Ax_1)^H x_2 = x_1^H A^H x_2 = x_1^H Ax_2 = \lambda_2 x_1^H x_2 = \lambda_2 \langle x_1, x_2 \rangle.$$

Since $\lambda_1 \neq \lambda_2$ it must be the case that $\langle x_1, x_2 \rangle = 0$. This shows that the eigenspaces of A are orthogonal.